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THE THIRD CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY.

The Third Conference of the Indian Mathematical Society was held at Lahore under the kind invitation of the Punjab University Mathematical Association on the 26th, 27th and 28th March 1921.

The opening ceremony on the first day was held at the Town Hall, and the meetings on the second and third days took place in the Law College Buildings, near the University Library, Lahore.

The Punjab University Mathematical Association, under the able guidance of their Patron, Sir H. J. Maynard, Vice-Chancellor of the Punjab University, their President, Prof. Devi Dayal, and other members of the Committee had done all that was possible to make the Conference a success. The arrangements for the comforts of guests were highly satisfactory.

The Conference commenced its proceedings at 11 A.M. on the first day in the Town Hall, in the presence of a large gathering consisting of members of the Society, members of the Punjab University Mathematical Association and a number of visitors—both official and non-official—interested in the objects of the Society. Some ladies also graced the occasion by their presence.

Punctually at the appointed time, His Excellency Sir Edward Maclagan, K.C.S.I., K.C.I.E., Governor of the Punjab, accompanied by his Military A. D. C., arrived, and was received on the steps of the Town Hall by the Hon'ble Sardar Bahadur Sunder Singh Majithia (Chairman of the Reception Committee), Mr. Balak Ram (President of the Society), Mr, Devi

Dayal (President of the Punjab University Mathematical Association) and members of the Managing Committee of the Indian Mathematical Society. After the formal introduction of the vsrious officials of the Society, His Excellency was conducted to the Conference Hall where the whole gathering received him standing. After His Excellency had taken his seat on the dais, the proceedings commenced.

The Chairman of the Reception Committee, the Hon'ble Sardar Bahadur Sunder Singh Majithia, Member, Executive Council, Punjab, first read his Address of Welcome.[1]

This was followed by the Opening Address[2] of His Excellency Sir Edward Maclagan, Governor of the Punjab.

The Hon. Joint Secretary (Prof. D. D. Kapadia) then read the Report [3] of the Society.

Mr. Balakram, M.A., I.C.S., afterwards delivered the Presidential Address.[4]

The Hon'ble Sir H. J. Maynard, K.C.I.E., I.C.S., Vice-Chancellor of the Punjab University, next proposed a hearty vote of thanks to His Excellency the Governor for the trouble taken by him to preside at the opening of the Conference and thereby contributing to the success of the Conference.

The proposal was received with acclamation, and the formal proceedings of the Conference for the day terminated.

After the departure of His Excellency and visitors, the members of the Indian Mathematical Society and of the Punjab University Mathematical Association met under the presidentship of Mr. Balakram and the following business was transacted :—

(1) It was proposed by Prof. P. V. Seshu Aiyar and seconded by Prof. G. S. Chowla—that the Society records with extreme regret its deep sense of the loss sustained by India in the sad and untimely death of the late Mr. S. Ramanujan, B.A., F.R.S., at the very early age of 32 years.

The proposal was received most solemnly, all the members standing.

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[1] Sardar Bahadur Sunder Singh Majithia's Address of Welcome is given on p. 90.

[2] The Opening Address of His Excellency is given on p. 92.

[3] The Hon. Secretary's Report is given on p. 93.

[4] Mr. Balakram's Address is given on p. 97.

(2) It was next proposed—that a befitting memorial, in a form to be determined by a Committee to be appointed hereafter, be instituted to perpetuate the memory of the late Mr. Ramanujan.

The proposal was moved by Prof. P. V. Seshu Aiyar and seconded and supported by Professors Samuel Lal and Mukundlal, and was carried unanimously.

During the discussion, that followed the proposal the general consensus of opinion of the members present seemed to be in favour of founding a prize to be awarded every year, or once in two years according to the funds available, in open competition throughout the whole of India, for original mathematical work.

(3) It was then proposed by Prof. P. V. Seshu Aiyar and seconded by Prof. Mehr Chand Suri—that a Committee, consisting of the following gentlemen, with power to add, be formed to give effect to the formulation of such a scheme to found a Ramanujan Memorial Prize :—

Messrs. Balakram, M. T. Naraniengar, P. V. Seshu Aiyar, R. Ramachandra Rao, G. S. Chowla, Hemraj and D. D. Kapadia.

The proposal was accepted unanimously.

(4) The discussion afterwards turned on the subject-matter of Mr. Balakram's letter for the improvement of the Journal. The main points of Mr. Balakram's letter were to issue our Journal in two parts—one containing mathematical notes, short articles, problems, &c., forming the elementary section and the other containing original advanced contributions.

In opening this subject for discussion, Mr. Balakram informed the audience, that the members of the Committee were generally sympathetic with his views but some had indicated the practical difficulties in the execution of such a scheme, and he wished to know the opinions of those present on these points. An interesting discussion then followed in which Messrs. Hemraj, P. V. Seshu Aiyar, M. T. Naraniengar, C. V. H. Rao, Samuel Lal and others took part.

In winding up the discussion Mr. Balakram suggested that as the matter had been sufficiently well discussed, it might now be left to the Managing Committee to go into the details and put forward definite proposals in the matter.

(5) Mr. P. V. Seshu Aiyar had issued a circular letter to the Committee for the more efficient working of the Library—his principal suggestion being to establish branch circulating libraries at various Provincial centres. In connection with this, the various replies which had been received were put before the meeting, and the long letter containing nearly all the details of our Library from our Assistant Librarian Prof. V. B. Naik was also read in part. General unanimity was not to be found and there was nearly equality of opinions for and against such branch Libraries.

The question of the efficient working of the Library has been a moot point ever since the foundation of the Society, and it has been found hard to arrive at a generally satisfactory solution of this problem.

After some further general discussion, the meeting terminated at 2 P.M.

The Hon'ble Sardar Bahadur Sunder Singh Majithia, Chairman of the Reception Committee, was "At Home" to members of the Indian Mathematical Society and of the Punjab University Mathematical Association in the evening on the same day at 5 P.M., at his residence, and the function was a grand success. The Hon'ble Sardar Bahadur spared no pains to look to the comforts of the guests and the cool and shady recesses of the spacious garden lawns added greatly to the enjoyment. The visitors regaled themselves till dusk and parted with happy memories of the Sardar's hospitality. This terminated the first day's programme.

On the second day, at about 8 A.M., the regular work of the Conference commenced in the Law College Buildings, near the University Library under the Chairmanship of Mr. Balakram, the President of the Society. The following papers were read, and the Conference adjourned for the next day at about 10 A.M. [The sitting had to be suspended for a few minutes at 9 A.M. for a group photo of the members of the Conference] :—

1. Mutually Locked Tetrahedra—by Prof. C. V. H. Rao.

2. Normals to an Algebraic Curve—by Prof. Hemraj.

3. Generating Regions of a Quadric in space of n-Dimensions—by Mr. R. Vythynathaswami.

4. An Elementary Exposition of the allied properties of the Simson Line, the Sternier Envelope, and the Orthopole—by Mr. A. A. Krishnaswami Aiyangar. [Read by Mr. R. Vythynathaswami, in the absence of the author.]

5. Note on the area of a Triangle circumscribed about a Parabola—by Mr. M. K. Kewalramani. [Read by Mr. R. Vythynathaswami, in the absence of the author.]

6. Convergents to $\sqrt{2}$ in the solution of some Pythagorean Equations—by Messrs. K. B. Madhava and Charles Saldhana.

In the afternoon, at about 4 P.M., the members were taken to Shahdara in carriages specially provided for the purpose, and had the splendid opportunity of seeing the historic monuments of the Mausoleums of Jehangir and Nur Jahan. After light refreshments and a stroll in the gardens, the party returned to Lahore at 6 P.M.

In the evening at 7 P.M., a popular lecture under the auspices of the Conference was delivered by Mr. T. P. Bhaskara Sastri of Nizamiah Observatory, Hyderabad (Deccan), in the Hall of the Society for the Promotion of Scientic Knowledge. The lecture was on "Star-Motion" and was illustrated by a number of lantern slides. Mr. Balakram, President of the Society, presided on the occasion, and the lecture was listened to with great attention by a crowded audience. After a hearty vote of thanks to the lecturer, the audience dispersed at about 8 P.M.

On the third and last day, the Conference met again at 8 A.M. and the following papers were read :—

7. Dr. Stewart's Theorems in Non-Euclidean Geometry—by Mr. M. K. Kewalramani. [Read by Mr. R. Vythynathaswami.]

8. Theory of the Rational Transformation—by Mr. R. Vythynathaswami.

[This paper is an attempt to form an integral conception of the general rational transformation in one variable in its most characteristic features. The positive results established are not many; in fact, more problems are raised than solved. In spite of its fragmentary character, the paper is believed to be of some value as it fixes certain general lines of enquiry in the subject of rational transformation.]

9. An Extension of Feuerbach's Theorem (Supplementary)—by Mr. M. Bhimasena Rao. [Read by Mr. K. B. Madhava.]

[Theorem : " Of the system of pedal-contact circles, each circle touches four circles, which latter are also touched by three more circles of the system."

The present paper contains the proof of the above theorem on the following lines :—

(1) Of the system of pedal circles of isogonal conjugate points whose join passes through a fixed point, each circle is touched by four and only four circles of the system.

(2) The common tangents at the points of contact touch a conic which is considered in some detail. This conic reduces in particular cases to the *ortho-polar conic* of a point.

(3) Each circle c_1 of the system is the pedal circle of a pair of isogonal conjugate points, say, (P_1, P'_1) and the four circles $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ which touch c_1 are the pedal circles of isogonal conjugate points, say (Q_r, Q'_r), $(r = 1, 2, 3, 4)$.

When the join of the isogonal conjugate points passes through the ortho-centre of the anti-medial triangle, the circles belong to the *pedal-contact* system. In this case, it is shown that the 8 Q-points form a pair of Steinerian quadruples.

It follows that the three remaining circles besides c_1 which touch a γ-circle are the pedal circles of points (P_r, P'_r), $(r = 2, 3, 4)$ forming with (P_1, P'_1) a pair of Steinerian quadruples.

(4) The three sides of a triangle and the nine-point circle may be taken to be c-circles when the γ-circles will be the in-and the ex-circles. This is the Feuerbach's theorem.

The relation between the P and Q points are under investigation.]

10. Mathematics in India, Then and Now—by Mr. V. Sankaran. [Extracts were read by Mr. D. D. Kapadia].

11. Some Diophantine Equations—by Mr. Balakram.

[Denoting by $f(a)$ the diophantine equation $x^2 - ny^2 = a$, where n is an integer without non-square factors, it is well-known that $f(1)$ can always be solved, and that a *necessary* (but not always *sufficient*) condition for $f(a)$ to have solutions is that $(-a)$ should be a quadratic residue of n. This note is written with the object of enquiring if the conditions *sufficient* to ensure the solution of $f(-1)$, $f(-2)$ and $f(+2)$ were known. It can be shown that if $n \neq 2$, not more than one of these equations has solutions; and that if n is an odd prime or twice an odd prime (not 2), at least one has a solution. The question " which one ? " can be completely answered by elementary considerations if n is odd ; or is even and of the

forms $2(8k+3)$, $2(8k+5)$, $2(8k+7)$. If $n = 2(8k+1)$, the following results are obtained :—

Put, as can always be done, $\frac{n}{2} = \alpha^2 + 16\beta^2$, α being odd. Then

$$\left.\begin{array}{l} \text{if } \beta \text{ is odd, and } \alpha = 8p \pm 1,\ f(+2) \text{ has solutions;} \\ \text{if } \beta \text{ is odd, and } \alpha = 8p \pm 3,\ f(-1) \text{ has solutions;} \\ \text{if } \beta \text{ is even, and } \alpha = 8p \pm 3,\ f(-2) \text{ has solutions.} \end{array}\right\}$$

The remaining case (β, even, $\alpha = 8p \pm 1$ remains unsolved. Calculation shows that out of 28 values of n of this type up to 7522, $f(-1)$ has solutions in 10 cases, $f(+2)$ in 10 cases, and $f(-2)$ in 8 cases.]

12. On a Special Cubic—by Mr. Hemraj.

13. On the Summation of Series of the Type

$$1 + \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{7^2} + \ldots\ldots$$

by Mr. C. Krishnamachari. [Read by Mr. G. S. Chowla.]

14. The Herpolhode Theorem—by Mr. S. R. Ranganathan. [Read by Mr. Balakram.]

[It is well-known that the motion of a rigid body fixed at a point O and acted on by no forces or by forces whose resultant pass through O, is equivalent to the rolling of the Ellipsoid of Inertia of the body at O on a plane known as the Invariable Plane.

The locus, on the plane of the point of contact of the ellipsoid is called the herpolhode.

Now, if P is the point of contact at any instant, it is also the point of intersection with the plane of the instantaneous axis of rotation, so that OP is the instantaneous axis of rotation. It is, further, easily shown that OP represents in magnitude and direction the instantaneous angular velocity of the body. Let us represent this vector by ω.

Then, P is a point on the herpolhode and the tangent at P to the herpolhode must be co-directional with $\dot{\omega}$. For, both the directions coincide with the limiting position of PP′, &c. ...

Further, if P is to be a point of inflection of the herpolhode, the direction of the tangent at P to the herpolhode must be stationary, *i.e.*, the change in the tangential vector $\dot{\omega}$ must be in its own direction, *i.e.*, $\dot{\omega}$ and $\ddot{\omega}$ must be co-directional or in vectorial language $V\,\dot{\omega}\,\ddot{\omega}$ must be zero. That this is impossible is easily derived from Euler's equations, as follows :

Now, the vectorial equivalent of Euler's equations

$$K\dot{\omega} = L - V\omega q$$

where L is the moment of the impressed forces about the fixed point O, q is the vector representing the moment of momentum of the system with reference to the point O, K is the linear vector operator or the inertia operator of the system with reference to O, representing the operations on ω indicated by the formula

$$K\omega = \omega\sum mr^2 - \sum m(\omega r)r$$

where the summation is extended over the whole of the rigid body and r is the vector connecting O to the element m.

K being independent of time and in the motion considered, L being zero, we have

$$K\dot{\omega} = -V\omega q = Vq\omega ;$$

so that

$$K\ddot{\omega} = \frac{d}{dt} Vq\omega$$

$$= Vq\dot{\omega}$$

since q is constant in the motion under consideration. Thus, it follows that $\ddot{\omega}$ is at right angles to the plane containing the vectors q and $\dot{\omega}$, so that it is at right angles to the vectors $\dot{\omega}$. Hence $\ddot{\omega}$ and $\dot{\omega}$ cannot be co-directional for any point of the herpolhode, that is, the herpolhode has no point of inflection.]

15. On some properties of Euler's and Prepared Bernoullian Numbers—by Messrs. C. Krishnamachari and M. Bhimasena Rau. [Read by Mr. K. B. Madhava].

[Form a Table thus :—

$1.n.$	$3n+2.$	$3n^2+2n.$	$15n^2+20n+16.$	$15n^3+20n^2+16n.$
$2(n+1).$	$6n+8.$	$12n^2+18n+16.$		
$3(n+2).$	$12n+20.$			
$4(n+3).$	$15n+40.$			
$5(n+4).$				
....				
....				
$r(n+r-1).$				

The second column is obtained from the first by an obvious addition,

$3n+2 = 1.n + 2(n+1)$; $6n+8 = 3n+2+3(n+2)$; &c.

The third column is obtained by multiplying the second column by the first. The fourth by addition from the third. The fifth by multiplying the fourth by the first column as before. And so on. Then, it is proved that the quantities at the top in the various odd columns are the numerators of the co-efficients of the various powers of x in the expansion of $\sec^n x$.

Ex. $n = 1$. From a table with $1^2, 2^2, 3^2, \ldots$ in the first column, we get Euler's numbers.

$n = 2$. We get the prepared Bernoullians, viz. $2^{2n}(2^{2n}-1)\,\frac{Bn}{2n}$, which are the co-efficients in $\sec^2 x \left(= \frac{d}{dx} \tan x\right)$.]

At the conclusion of the reading of the papers, Prof. Devi Dayal, President of the Punjab University Mathematical Association, proposed that the best thanks of this Joint Sessions of the Indian Mathematical Society and the Punjab University Mathematical Association are due to His Excellency the Governor, the Vice-Chancellor, the University authorities, and the Hon'ble Members and Ministers for their sympathy, co-operation and financial help. Prof. Devi Dayal put this proposition before the audience in a short felicitous speech[1] and on being seconded by Mr. Balakram, it was carried with acclamation.

It was next proposed by Mr. Balakram and seconded by Prof. P. V. Seshu Aiyar that the best thanks of the Indian Mathematical Society are due to the Punjab University Mathematical Association for the generous manner in which they had taken upon themselves the onerous task of allowing the Society to hold so successfully its Conference at Lahore. Also similar thanks are due to His Excellency the Governor, the Vice-Chancellor, the Members of Council, and the Ministers of the Province for their kind co-operation in this matter; also to Profs. Devi Dayal, Das Gupta, Hemraj, Mehr Chand Suri, and to students and volunteers for their selfless help, without which the Society would not have been able to hold its Conference at Lahore. The proposition was carried unanimously.

Finally, with a hearty vote of thanks to the Chair, proposed by Prof. Devi Dayal and seconded by Prof D. D. Kapadia, the proceedings terminated.

In the afternoon at about 3 P.M., the visitors were taken over to the North-Western Railway's Workshops, nearly the biggest workshops in the whole of India. The inspection of these workshops on the extensive railway grounds was somewhat tiresome, but nevertheless it was most instructive and interesting. After mutual greetings, and hopes of meeting again on similar occasions at future conferences, members bade good-bye to one another.

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[1] Prof. Devi Dayal's speech is given on p. 108.

Address of Welcome by the Hon'ble Sirdar Bahadur Sundar Singh Majithia.

"Your Excellency: When in the unavoidable absence of the Hon'ble Khan Bahadur Mian Fazl-i-Hussain, Minister of Elucation, Punjab, I was approached by my friends to take up the happy duties of welcoming Your Excellency and the distinguished gentlemen who come from such long distances to attend to-day's Conference of the Indian Mathematical Society, I was assured that no difficult sum would be set before me to do. I explained to my friends that when I left the Government College in 1893, I did so with a sigh of relief; not that I disliked staying in that institution where I had spent two happy years of my life, but on account of the subject which will be deliberated upon to-day by the present day Indian savants in that science. In the first few months of my College life I had the honour to sit at the feet of the well-known Babu S. B. Mukerji, a mathematician of some fame. I must frankly admit that he was beyond my poor grasp. He was a brilliant mathematician and it was hard for a young student like myself to scale the giddy heights to which he jumped omitting the intermediate steps without which it is hard for a beginner to proceed. Then I was placed in charge of the well-known Professor Mr. G. N. Chatterjee who wished his students to follow him with the speed of a mail train. I am not in any way speaking with disrespect of my revered Professors but of the difficulty that I personally had to contend with. The courses were finished in the first three months and then we were regaled with happy and pleasant stories of his wide travels which kept us enchanted and the pleasant memories of the period I find it hard to forget. Of course there was a revision of the subjects later on. To the above was added my school boys dread of the subject. So I am sure Your Excellency and the other distinguished gentlemen present here to-day will forgive me if, at the outset, I say that I find myself in an anomalous position as regards the subject-matter of to-day's Conference. In my final examination when the time drew near for me to sit for my Intermediate test, I was not at all sanguine of success and as I did not want to extend the period of my Court of Wards' supervision I approached my respected Professor, Mr. Chatterjee, for advice. He always found some of us sitting at our fixed places as permanent members of what he called the bench of honour. I put my difficulties before him and asked him how I could pass the examination. He asked me if I could cram up some portions of Euclid, Trigonometry and Conic sections. I told him I had no taste for Algebra but would do my best in the above subjects to carry out his kind advice. He in his humorous vein advised me to throw my Algebra book in the *chohla* and this I confess I did though not by actually throwing the book in the first place but by laying it aside never to be

opened again. I could do a little of Arithmetic, the additions and subtractions which came in very handy when I was Honorary Secretary of the Khalsa College and also in keeping my private accounts. In my Matriculation, I knew the four books of Euclid by heart. I could repeat them from beginning to end and I had not forgotten these. The remaining two or three books—I don't remember now how many they were—I also committed to memory. I still remember the formula of $\sin(A+B) = \sin A \cos B + \cos A \sin B$, in Trigonometry. I confess I only remember by cram. I hope I have remembered and expressed it aright.

If not, then the distinguished audience will excuse me as I can assure them that since closing my book with a sigh of relief in 1893, I have not had an occasion to open it. My inability to grasp the subject lay in my own weakness: as a boy I was not very much attached to my subject and in my school days it was not so lucidly explained as I saw it explained when I was Honorary Secretary of the Khalsa College by Professor Nihal Singh. If it had been taught me in school in that way, my aversion to Algebra might not have been there. And I was not able to climb the lofty heights of Professor Mukerjee, or to keep pace with the mail train speed of Mr. Chatterjee—perhaps owing to my dulness in the subject.

All the same, I admit that the science of Mathematics is one which could not be ignored. Even the world is running smoothly in its mathematical precision in the courses chalked out or ordained for it by the Almighty. But for this, the land of Five Rivers, the home of the Rishis of old, would not produce the abundance of wheat which feeds the world. With the aid of this science, the Punjab rivers have been harnessed and the jungle tracts—once the habitation of the King of Forests—and the barren Bars of Lyallpur and Montgomery, the latter being called the Ganji Bar and where antelopes even died of thirst, are teeming with green fields all around. So I must say that though I am unfortunate in not having benefited myself and taken advantage of the opportunities that were afforded me in my College days in the personalities of the two eminent Professors at whose feet I had the honour to sit, but still I consider the subject one that we could not ignore. Your Excellency, we have many a difficult problem to solve these days and it would be out of place for me to mention those. But if the deliberations of my talented friends to-day solve the questions they have set before themselves correctly, which I have no doubt they will do, they will be adding to the knowledge of the world in a subject which I may safely say is indispensable to mankind. Let me further hope that my friends will apply their store of knowledge to the finding of a solution which will bring them nearer to their Maker,

If one could only have a ray of light from that store of all knowledge, the life in this toiling and moiling world will be a happy and a contented one and realising our relationship of the Fatherhood of God and the Brotherhood of Man, we shall lead a happy and a graceful life in this world.

Let me once more offer Your Excellency and my talented friends gathered together in this Hall from the four corners of my country, my most sincere and warm welcome, in the full hope that they will solve the problem of life successfully. I would now request Your Excellency to be so good as to declare the Conference open."

The Opening Address of His Excellency Sir Edward Douglas Maclagan, K.C.S.I., Governor of the Punjab.

"I need hardly say, gentlemen, that I greet the Members of this Conference with much warmth and I may add that I also do so with much awe. I am aware that the gentlemen who tried to teach mathematics in my youth would be much shocked if they could learn that I had attempted even to approach the threshold of a learned Conference such as this. I do not wish to discourage the examination of candidates for the public service in mathematics, but I am free to confess that I have myself managed to get through a longish service without using any other mathematics than arithmetic. I was indeed once ordered as a Census Officer to use a formula which my friends told me entailed the use of the Differential Calculus. I think it was to ascertain the average distance of one Post Office from another, or some similar useful information, and I am afraid that I was reduced to reporting somewhat evasively that the average distance was not very different to what it had been before. I am proud, however, to say that our services in the Province have from time to time included some admirable mathematicians, more especially among that fine body of Engineers who construct and maintain the celebrated system of Punjab Canals. We have also had mathematicians in the Civil Service, and we are just now arranging to give facilities to one of our best mathematicians in that service to devote some time to the scientific examination of the forecasting of crops.

"I am glad to see here to-day not a few of the best known mathematicians in India, and I feel that a great honour has been conferred on the city of Lahore by their visit. I do not know whether this part of India had any special claims to mathematical distinction before the era of Prof. Chowla and before we started the Honour Schools of Mathematics under a Professor with a European reputation. We have all of us however heard something of the long and honoured career of

Mathematics in India generally. I notice that a recent writer on exact science in India is of opinion that the pure mathematics of the ancient Hindus was on the whole not only in advance of the Greeks but anticipated in some remarkable instances the European discoveries of the 16th and 18th centuries. The origin of the decimal system is attributed to them; and they are said to be the real inventors of Algebra, and I am told that some very early Indian mathematical treatises contain spirited attempts to square the circle. With these historical foundations stretching so far below him, it is no wonder that the modern mathematician in this country has risen so high.

" We have not so long ago lost that brilliant genius, Mr. Ramanujan, but we have I hope others of similar calibre still among us, possibly among us here to-day.

" With the spirit of Brahmagupta and Bhaskara behind him, as well as the teachings of Laplace and Newton, the modern Indian mathematician has a start which he will doubtless worthily maintain, and I hope that the distinguished gentlemen who have met here to-day will by their mutual intercourse intensify the favour and success of studies in India.

" I could wish too that you would leave some of your enthusiasm behind you for our beneft in the Punjab. Speaking as a mere administrator, I am bound to say that our most crying practical need here is something far beneath your notice, namely, an improved knowledge by our peasants of the methods of calculating compound interest, but there are others and larger spheres of practical and intellectual value in which I trust that your presence among us may be found of much real use to the Province.

" I shall not detain you longer, gentlemen, and wish you every success in your Conference."

Report of the Hon. Jt. Secretary, Mr. D .D. Kapadia.

" Your Excellency and Gentlemen—

On behalf of the Committee of the Indian Mathematical Conference, it is my pleasant duty to extend to you all a very hearty and cordial welcome on the occasion of this—the Third Conference of the Indian Mathematical Society—and to lay before you a brief report of the work done by the Society during the last two years. The Society cannot but feel extremely thankful to you, both members and well-wishers, for having taken the trouble to be present at the Conference and contributing your individual quota to make it a success.

We consider ourselves extremely fortunate that on each occasion on which we held a Conference we have been uniformly successful in securing the kind and patronising co-operation of the Governors of the Provinces within whose jurisdiction we had the good fortune to meet. His Excellency Lord Pentland opened our First Conference at Madras in December 1916: His Excellency Sir George Lloyd was to open our Bombay Conference in Bombay, but owing to His Excellency's unavoidable absence, at his own express wish, Sir George Carmichael, the Senior Member of the Bombay Executive Council, performed the opening ceremony in January 1919, and to-day on the occasion of the Third Conference, we have been fortunate in having Your Excellency amongst us to perform the opening ceremony and set the seal of approval of the Government of the Punjab on the humble work which our Society has been trying to do for the last 14 years. Your Excellency's presence amongst us to-day will inevitably act as a strong stimulant to our efforts; and our Society cannot but feel highly thankful to Your Excellency for this kind condescension on your part.

When our Society was founded in 1907, it was but a very remote possibility to hold conferences. With great hesitation and diffidence, the first experiment was tried at Madras. The experiment turned out to be so very satisfactory, that two years later the Second Conference was held at Bombay, and after another two years we meet here to-day at Lahore. We feel very grateful to the Punjab University Mathematical Association for giving us this opportunity to meet here, as the Association have been doing the utmost in their power to make this Conference a success.

In the last report of the Committee before the Bombay Conference, the Committee had expressed themselves in some such words as these, *viz.*, "the dire war has now ended and peace is within sight and the humanity will soon be again taking a breath of relief. People will now have leisure and energy to devote to their domestic problems, and our Committee hoped to achieve some tangible results in the near future." These were our pious wishes. But the Committee regret very much to say that they have not been able to do what they so sanguinely hoped to achieve. The world-wide after-war reactionary movements had their dire effect even on the working of our peaceful Society, and beyond doing the actual routine work which the Society had been doing previously, nothing new of any very striking importance could be done during the last two years. Owing to printing difficulties, our journal (which is as some members are pleased to say, the living monument of the existence of our Society), had to go through several vicissitudes, and had it not been for the indefatigable exertions of our Editors, Messrs. M. T. Naraniengar and P. V. Seshu

Aiyar, it would have remained far more in arrears than what it was at present. With the beginning of this year, we have entered into the 15th year of our existence, and our journal commences its thirteenth volume.

Whsn we met last, the strength of the Society was 197 members—viz., 1 patron, 1 hon. member, 9 life members and 186 ordinary members. To-day our net strength is 215 members, viz., 1 patron, 1 hon. member, 11 life members and 202 ordinary members.

During the interval of two years, we have to record with regret among others, the sad death of H. H. the Maharaja Bhawinghji of Bhavnagar and Mr. S. Ramanujam, F. R. S. His late Highness the Maharaja of Bhavangar was along with His late Highness the Maharaja of Mourbhanj Baripàda, one of the foremost life-members of our Society, and in the sad and untimely death of the late Mr. S. Ramanujan, F. R. S., at the early age of 32, India has lost a most talented Mathematician, whose place is very difficult to be filled up. Shortly after the late Mr. Rama. nujan's return from England in April 1919, our Society elected him as an Hon. Member, but to the bitter disappoinment and regret of the whole of the scientific world, he breathed his last in April 1920, and to-day our Society is going to discuss a proposal to commemorate his memory. But whatever the outcome of that proposal is going to be, the name of Ramanujan shall ever be cherished fondly by India and his loss shall ever be deeply mourned.

In the previous report, the Committee have already touched upon the important essentials wanting both in connection with our Journal and our Library and hence it is not thought advisable to speak at length on those subjects. The Society is well cognisant of its deficiencies in these directions. To-day immediately after the formal Sessions of our Conference are over, there will be a discussion at our general meeting of the various suggestions put forward by our President, Mr. Balak Ram, for the general improvement of the get-up of our journal and the subject-matter for our journal. It is suggested that our journal should be published in two parts—one Elementary, for the ordinary students of Mathematics, and the other Advanced for research scholars. When our Society was started in 1907, there was no other society of a similar type in existence in India. Following in its wake, the Calcutta Mathematical Society was started, and recently the Benares Mathematical Society was started at Benares. Also the Punjab University Mathematical Association, as also a similar Association at Madras are now in existence. The number of Indians with European qualifications in Mathematics, as well as the number of students going to Cambridge and Europe and America was then limited. But now the number is steadily increasing; and suffice it to say

that talented Indians are now taking a more vivid and keener interest in mathematics than was possible in those days and hence the original lines on which the Society has been doing its work may require some modifications and we are going to discuss to-day the ways and means by which the objects for which the Society has been originally started can be achieved better in the present times.

Our Library has also its similar deficiencies and the number of books in our Library is only 500, and the number of volumes of Mathematical Periodicals is only 375. The Library receives nearly 30 mathematical magazines from Europe and America mostly in single copies, and at present there are nearly on an average 22 demands for each magazine. If then each member passes on the magazine after keeping it in his possession only for 10 days, the last member would receive it after 210 days? But human nature is weak and prone to err and it is found on inquiries that hardly 10 per cent. of the magazines issued to the members during the last two years have yet come back to the Library. There are some very interesting statistics and the report of the working of the Library submitted by our Librarian, will also be placed before our meeting for information. Suffice it to say that our Library is suffering both for want of funds and for want of active co-operation on the part of our members. Unless we canvass sufficient funds, we shall not be able to buy back numbers of at least some of the important mathematical journals and also a good number of volumes on Advanced Mathematics facilitating mathematical research. Also to ensure quick circulation of current magazines we badly want some important journals in duplicate or more copies. In connection with this, our Committee feel great pleasure in mentioning that our present President, Mr. Balakram, has lately given us a handsome donation of Rs. 1,000 to have back numbers of some of the important journals and the best thanks of the Committee are due to him for the same. At the same time the Committee most earnestly hope that under the present regime, the Indian Ministers in charge of the Departments of Education in the various provinces will not fail to extend their help in the form of library grants to our Society and enable us to fulfil the objects which we have so fondly cherished up to now. To achieve these objects the Committee expect the very active and enthusiastic co-operation of all, both the old and the young members of the Society, more especially the latter on whom the future regeneration of India so intimately depends. The Committee, therefore, take this opportunity to make this strong appeal not only to the members of the Indian Mathematical Society but to all those patriotic Indians interested in the science of mathematics—one of the most ancient and the most useful of the sciences—to contribute their mite towards

the fulfilment of the objects for which the Society was primarily founded.

In conclusion, the Committee once again take this opportunity to record their high sense of appreciation for the trouble taken by Your Excellency to preside on this occasion, and to partake in the deliberations of this Conference in the midst of a multitude of pressing engagements. The Committee also take this opportunity to offer their best thanks to all those who have been ungrudgingly contributing their mite for the welfare of the Society, during these 14 years of its existence. I is also a very gratifying feature to note that some of our members have taken the trouble to undergo the fatigue of a tiresome journey of some 2900 miles to attend this Conference. The Committee take this opportunity to thank cordially these and all others who have worked so selflessly for the successful holding of this Conference at Lahore."

The Presidential Address by Mr. Balak Ram, M.A., I.C.S.

Mr. Balak Ram, M. A., I. C. S., President of the Indian Mathematical Society, delivered the following Presidential Address :

Gentlemen,—Our Society includes among its members a large number of men who were eager students of mathematics in their early days, but whose other occupations leave them hardly any leisure for keeping up their studies. They, however, retain their interest in the subject because of the pleasure they derived and still derive from it, and they are willing to do what little they can to help the Society, feeling honoured when the professional mathematicians among you appreciate their efforts, as for instance, by electing one of them as their President. But, however honoured I personally feel, I confess that I would have felt greater satisfaction if we had been able to carry out our original intention to elect as our President for the current term, Pofessor K. J. Sanjana, who has been one of the most energetic of our members since the Society was started fourteen years ago. We deplore his inability to accept office and trust he will soon be restored to health.

Another stalwart whom we miss to-day is Prof. A. C. L. Wilkinson, my immediate predecessor in office. He presided at the First and Second Conferences of the Society and he has given us freely of his experience and knowledge, first as a member of the Committee and later as President for four years from December 1916.

During the last year, we have lost the active co-operation, due to departure from the country, of Sir John Heaton. Those of us who attended the Conference in Bombay in January 1919, will agree with me that the success of that meeting was largely due to his efforts as the Chairman of the Reception Committee. Sir John Heaton was a member of the Indian Civil Service for nearly 37 years and rose to be a High Court Judge, which post he held for nearly 12 years before retirement in April 1920. At one time he was Vice-Chancellor of the Bombay University on which body he did useful work. His keen analytical mind, his deep and sympathetic knowledge of human nature, his complete freedom from communal and individual prejudice made him a great judge, and his capacity for loyal friendship and wise guidance won for him the affection and respect of all who knew him. He came to India at an early age, as was customary in the eighties, and never had an opportunity of going through a full course of mathematics at a University; but mathematics always fascinated him and he joined our Society on his own initiative as soon as he heard of it. We could always count upon him in time of need. We wish him peace and happiness in his retirement.

Another gentleman whom I mention by name is the Hon'ble Mr. R. P. Paranjpye, our first Honorary Member and our Librarian, who has been prevented from attending the Conference by pressure of his new duties. We congratulate him on being in a position where his powerful personality and abilities will have fuller scope than hitherto. His general career is well-known to all of you, but most people are not aware of one of his activities in pushing forward a scheme which I believe is destined to have an important effect on the future of this country. I refer to his advocacy of the method of Proportionate Representation to provide for communal and minority representation on public bodies. In India, the problem is acute; but the compromise solutions accepted so far are, I am afraid, neither acute nor right, but quite obtuse. It is interesting to note that Mr. Paranjpye's first observations on the subject were published as a short note in our Journal, under the heading "An Electoral Anomaly". After some years of study, he came to the conclusion that the system prevalent in several places in Europe, America and Australia, of Proportionate Representation with a single Transferable Vote gave as satisfactory a solution as was likely to be obtained as long as man was what he is. The working of the scheme is simple as far as the voter is concerned, but rather complicated when the votes are to be counted. A few weeks ago the scheme was adopted by the Council of State for election to their Committee and I have no doubt that one day it will be adopted for all purposes by all elective bodies. But when the scheme was first brought up before

one of the provincial legislatures three years ago, it was rejected with scorn, one indignant opponent calling it "Mr. Paranjpye's Mathematics". So I take it that the final acceptance of the scheme may be legitimately put on the credit side of mathematical ideas as supplied to administrative problems.

Probably some of the audience will be surprised to hear mathematics and administration mentioned together. It is true that the higher development of mathematics does not solve current difficulties of administration; but the same remark applies to ancient history, archæology, and most other branches of knowledge. What is useful here is the particular outlook which the study of science (especially mathematics) tends to produce with peculiar intensity,—a clear enunciation of the fundamental principles, a reduction of the number of those principles to a minimum, exact thinking, and a quantitative appreciation of the degree of exactitude attained by the accepted tentative solutions. These qualities make for success in all spheres; it is, therefore, not surprising that men with sound mathematical training have left their mark on non-mathematical subjects, such as philosophy and law. The contributions of mathematicians to the study of sociology, economics, and biological variations and heredity have been more direct, because these subjects deal with material in the mass and the science of statistics, (which is only another name for the mathematical theory of probability), deals with material in the mass. We all know the scoffer's jeer that statistics will prove anything. The fact is: *in the hands of an ignorant or an unscrupulous man* statistics *will* prove anything; just as in the hands of an ignorant or an unscrupulous man a surgeon's knife will kill anybody. Nor is there more truth in the celebrated classifications: "white lie, lie, statistics." In the hands of an expert, statistics lead to the discovery of unexpectedly accurate laws. An example probably known to you is the working formula obtained by Mr. Jacob, I.C.S., of the Punjab, for predicting the yield of crops. The most striking instance of the power of science in disentangling truth is furnished by Karl Pearson's analysis of mortality figures. He first determined the number of deaths in old age, in middle age, in youth and in childhood, and found them to follow the known laws of distribution of chance events. Eliminating these numbers, he was finally left with the curve of deaths in infancy. The curve was mathematically incomplete as if the initial part of it had been torn off. When completed algebraically, it was found to end at minus 9 months. The inference is that humn life commences normally 9 months before civic birth,—a conclusion obvious enough when stated; but the point is that it was arrived at by practically sitting at the cemetery gate, and doing nothing but noting the ages of the dead people. Those who are interested in this investigation will find

full details of the method and the results in Karl Pearson's lecture on "Chances of Death" in his book of that name. The generally accepted opinion is that the only reliable method of finding out laws governing social phenomena is the collection and analysis of statistics. I venture to commend this view to the able administrators who are present here to-day.

Since we met last, we have lost our most prominent member and one of the greatest Indians of modern times. I refer to Srinivasa Ramanujan, whose death at the early age of 32 is mourned by the whole mathematical world. Details of his uneventful life and expert appreciation of his work have been published in scientific and other journals, and are probably known to you. To-day I need only express the unanimous opinion of the Society that we all loved Ramanujan and that India owes a deep debt of gratitude to Professors Hardy and Neville for generously and ably befriending him at a critical time of his career. The Society will, I understand, be asked at the business meeting to-day to consider the question of perpetuating his memory in some form.

Mention of Ramanujan's name suggests the question of the organization and endowment of scientific research in India. Now that the official educational policy is in the hands of Indians and there is an increasing number of eager and capable men anxious to make educational experiments outside the official bounds, it will be our own fault if we do not make our educational system as efficient as that of other countries. Progress has been made in technical education, and of late there has been talk of research institutions, but the research that is attended to is such as is of immediate practical value, which means research which pays dividends. Very few people will deny the claims of technical education; we all want physical comfort; we much prefer travelling in a luxuriously fitted motor car on a smooth dustless road to travelling in what the municipality humorously calls a third-class hackney carriage on a *kutcha* path; we also want the best beloved to wear pretty silks; and if the car is not comfortable enough or the silk not pretty enough, we are willing to set up institutions for improving mechanical engineering, sericulture and the dye industry. This as it should be in the beginning; but it is to be remembered that technical education provides the pupil with a direct means of livelihood. If the trained student becomes a simple manufacturer, he makes goods of the old pattern and sells them; if he, like Edison, is more intellectually inclined, he becomes an inventor and sells his inventions; in both cases, his food problem is solved, and with luck he may even become passing rich on £40,000 a year, if not a millionaire. What I wish to emphasize is that the worldly rewards manufacturers and inventors receive are a sufficient incentive to the pupils of technical institutions and

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that after the initial stage is passed, the persons directly benefited by that education may well be required to pay for their training. The very different results following the pursuit of pure learning are vividly described in Browning's poem "The Grammarian's Funeral." There is no money to be earned, and the audience may be fit, but it is certainly "few," and not infrequently the work of a whole lifetime fails to score positive results. The desire for knowledge is, however, one of the primary instincts of the mind; therefore, we find scholars in all settled communities, men like Archimedes and Newton, the stories of whose absorption in their studies have been a tonic to generations of humbler students. Again, every settled community is sub-consciously or consciously convinced that attempts at progress are apt to be wasteful unless guided by thinkers and teachers who have leisure to think and teach clearly; therefore, we find these communities giving special protection and advice to learned men and thus providing them with leisure and the psychological incentives which replace the missing pecuniary reward. I speak of settled communities; but the most telling instance of this view-point is furnished by the rude barbarian Timur, about whom it is related that, when camping, the learned men (without whom he never travelled) were placed in positions of maximum security, while even the women and children had to be content with a less safe place. I do not know if Timur had a coherent theory behind his actions; if he had, he never explained it; he was not of that sort; but the modern man whose attitude towards learning is the opposite of Timur's and is symbolically represented by his thinking that Reading, Writing, and Arithmetic can be described by their initial letters as Three R's, makes no secret of *his* theory. "Research," he says, "is waste of time. Here I am, making piles of money out of sugar or diamonds or musk or whatever it is. What use is it to me, to learn about the stars or electrons or the differential calculus? Look at Professor K. Ley. He is bald and short-sighted and absent-minded, and his socks never match his ties. This comes of algebra and euclid and the fourth dimension." After this he talks patronizingly of schools and says that it is enough if a person is practical and has horse sense; and he further wants to know what those dashed dots are for.

Fortunately, power is passing out of the hands of this type of thinker and it is no longer necessary to explain that scientific research has a practical bearing. Difficulty, however, is experienced when mathematical research comes to be discussed, but there is sufficient evidence to show that though most results in pure mathematics are as a rule not applicable to any purpose of practical value, the work of a pure mathematician has frequently come to the help of physicists and others at unexpected moments. Students of probability know that the science of statistics to

which I have already referred, grew out of an academic study by pure mathematicians of games of chance. The laying of the Atlantic cable, it is said, would have been abandoned as an unprofitable task, if Lord Kelvin had not been able to solve a particular differential equation. Heavyside's skill in solving another differential equation enabled him to surmount a serious difficulty in long distance telephony. The latest instance of an apparently aimless research proving handy for physicists is furnished by Einstein's reconstructing the universe by utilizing Riemann's discoveries in non-euclidean geometry. I believe it is now generally admitted that during the late war pure mathematicians were able to render invaluable help in the development of aircraft, gunnery, and other matters. Instances might be multiplied; as a consolidated argument in favour of the proposition that research pays directly or indirectly, I refer to the example of the hard-headed practical American nation endowing professorships for theoretical research in such unpractical subjects as the Theory of Numbers.

This is our appeal to the moneyed man for endowment of mathematical research. To the mathematical student the study of the subject itself transcends in interest all practical applications. He studies the subject for the sake of the artistic pleasure he obtains in following its rigorous logic and in contemplating the extreme universality of the formulæ. Experience of life or nature gives us the data which the mathematician, instead of writing in English or Hindi, translates into his universally accepted symbols; but as soon as he has finished the work of writing down the symbols, he ignores for the moment their original significance and concerns himself with logical manipulation only. When he gets his final results, he translates them back into the language of the original data, leaving the physicist or the statistician or the engineer to verify the correctness of the data by further experiment. The mathematician is not concerned with the collection or the correctness of data. This aspect of the subject had led our most famous mathematical philosopher, Bertrand Russel, to describe mathematics as a science in which we neither know what we are talking about nor care whether what we say is true or not. The charm of mathematics lies in the fact that the description is accurate.

Whatever the point of view be, I earnestly plead for endowment of research in mathematics and in other sciences. I do not imagine that we shall discover a Ramanujan every decade; but I am sure there are enough people in the country who are willing to follow a career of quiet study and do respectable, if not highly distinguished, work, provided their minds are free from anxiety about the daily bread. I have only to mention the names of Bose, Roy; and (coming to younger men)

of Ganesh Prasad, V. Ramaswami Iyer, C. V. Raman, C. V. H. Rao, Birbal Sahni and Shivram Kashyap, to prove my assertion that there is indeed very promising human material in the country for work in all branches of science. Full opportunity alone is lacking.

I may take the liberty of giving a few suggestions of mine on the subject, not as an elaborated scheme, but as a basis of discussion. In the first place, it has to be noted that the research man cannot work to order any more than a Poet Laureate, even though fortified by an annual cask of wine. You must give the research man as much time as he wants and takes, and also must avoid as far as possible setting him a definite task. He has been well described as nature's loafer, he will neither be hurried nor commandeered, except in times of severe stress, as for instance during the Siege of Syracuse or the late war; and even then you do not get the best out of him unless you leave him alone. It is exceedingly improbable that Newton would have discovered the law of gravitation if he had been placed on special duty, with the usual deputation allowance of 2 %, to study the solar system within six months. Another point to be noted is that the research-man is human and has human needs and that you interfere with his efficiency if you do not provide him with a little comfort and free him from anxiety about the future of himself and his family. In addition to conducting his own research, he may be researchably required to discover his successors. For this purpose, he must be in constant touch with a number of selected students. The scheme that is suggested by these principles is that there should be research scholarships for the beginner and lectureships on a small pay for the student who has proved his worth; the lectureships should not belong to a graded service where the question of seniority or extension of service will cause heart-burnings and depression of mind; and their number should be large and variable, to provide for the appointment of young men without making the older men retire too early. To minimise distractions, there should be a fairly generous provision for individual family pensions and as an additional concession the children of research lecturers of a certain standing may be declared to be entitled to free education up to the highest standard for a certain specified number of years. The duties of the lecturer should be mainly research and control of research by the students, but he should be required to do routine teaching for two or three hours a week, as this would enable him to pick out likely men at an early stage. To allow the students themselves to focus their attention on research, the rules for the M. A. and higher examinations should be made more elastic. Finally, good libraries should be provided with complete files of all the leading scientific journals, and occasional visits by the lecturer to Europe and America should be encouraged.

The above is meant to be a rough sketch only; but it seems to me that anything worked on narrower lines will not give satisfactory results. And even then, it will not give sixteen annas worth for every rupee we spend, if the foundations of education are not properly laid in the school and the intermediate classes. I invite your attention to this urgent problem, and hope you with your experience of school teaching will devote your energies to solving it. The object to be attained is that the logical principles of the subject should be clearly taught to suitably selected boys and habits of clear and concise reasoning inculcated in all boys. Unless we do this, the educational value of mathematics in building up character will be lost.

Considering the wide interest taken by scientists and by laymen in EINSTEIN'S Theory of Relativity, it would not be out of place in a popular address to give a brief exposition of these speculations. The broad results of the theory are not essentially hard to understand; the difficulty, which is insuperable in many cases, consists in removing the old ideas firmly embedded in our minds in our younger days. It is not easy for any but the trained mathematician to realize that a euclidean straight line has no existence outside our conceptual world, and that there is no logical inconsistency in defining straight lines in such a manner that two straight lines can meet in two or three or a hundred distinct points. The euclidean straight line is simpler to argue about; but neither of them is really more straight than the other. Our children, trained differently will see this easily, as they will see that two and two are not alweys four, and will laugh at us as we laugh at our ancestors who thought the earth was flat. We, handicapped by our old notions, have to do the best we can by digging up the foundations of our mathematical beliefs and putting in as much of the new conceptions as we can. I therefore begin at the beginning.

Our knowledge of the world outside us is built out of our detached sense impressions, and the need for co-ordinating the knowledge establishing a chain of cause and effect arises because of the necessity of self-preservation; for instance, the organism that fails to establish the connection between the sensation of hunger and the effects of eating will die soon. The number of sense-impressions is large; our minds are finite; hence it soon becomes necessary to invent some shorthand formula for economising thought, each formula expressing some relation between two events (*e.g.*, the relation of before-and-after, here-and-there) in terms of an imagined entity (*e.g.*, time and space). When such entities become very well known, the human mind tends to endow them with objective reality. Philosophers may fight about the objective reality of time and space, but

when it comes to describing events or natural phenomena, we quietly ignore these controversies and argue as if time and space had an existence independent of the observer or the group of observers. Hence it came about that geometry was considered to embody absolute truth and many a weary soul found consolation in the thought that even on the farthest stars the three angles of a triangle are equal to two right angles. We have broken away from this faith. We know that geometry *as such*, being only pure mathematics, has no concern with absolute truths, and that the practical correctness of the conclusions is to be verified by the physicist who supplied the geometrician with the data (or postulates), such as the postulate that two intersecting straight lines cannot be parallel to the same line. If further experience of the universe gives a different set of postulates, the geometry that deals with them is different, and the conclusions are also different. It is the physicist who has to decide what sort of geometry best describes the universe. Let us consider the case of a race of beings living on this earth, with senses sharp enough to *see* (with the eye) the variations in the curvature of the surface in a small area but not possessing a worked out geometry. They will draw the shortest distances between two points and call them straight lines, and they will find that no two shortest distances if continued, remain a constant distance apart. They will also find things which fit on the surface of the earth in a smooth place will not fit in a rugged place. Their conception of space will be that it is not flat; and their geometry, when developed from these and other postulates will be a complicated non-euclidean geometry. They may, by a process of abstraction, be able to imagine a flat euclidean plane: but their space of experience (their space of physics) will be a non-euclidean plane.

This example will show from analogy that there is no reason why our space of physics should be euclidean. The difficulties created by electrodynamical phenomena led Einstein to enunciate a theory which was subsequently replaced by a more general theory in which the universe (consisting of space and time) is taken to be non-euclidean. This conception enabled him to dispense with the force of gravitation and to describe the movements of matter as always taking place along lines which were essentially straight; only the type of line which was straight in one place was not necessarily straight at another, as "matter" was nothing but a variation in space curvature altering the type of straightness in its neighbourhood. As a very crude and incomplete analogue in two dimensions of the theory, we may take a wash-hand basin with a sink to let out water. If we think of nothing in the universe except the wash-hand basin and the movement of drops of water in it, the sink will be like a centre of attraction and a drop of water placed on the inner surface of the basin will run to the sink along a shortest line, *i.e.*, along a straight line. But the drop of water will pass through the sink; and here the analogy fails as a particle of matter cannot pass through another.

The universe as imagined in Einstein's theory has four dimensions; and this is the second stumbling block in the path of the layman. The space of our everyday experience has three dimensions: length, breadth, and thickness, and we cannot frame a mental picture of a fourth dimension. To understand the theory, let us again go back to our sense-impressions which form the elements of our knowledge. We notice that every object (real or imaginary) has an existence in space and time, that is to say, it has extension (length, breadth and thickness or height) and duration, in all *four* variables. The mathematician expresses it by saying that every object, when its whole history is taken into account, has *four* dimensions; consequently the statement that the universe is four dimensional is only a new way of expressing a fact known even to cats and dogs and most of the higher animals. But—and here we come to the greatest discovery that the human race has ever made—the time dimension is essentially of the same sort as the three space dimensions, and the four are interchangeable, one man's time being compounded of another man's space and time. The three dimensional analogy is that the separate measures of the three space dimensions of a body change without the body changing in shape or volume. A brick placed on one face, and measuring one inch in the vertical direction, two inches in the northern direction, three inches in the eastern direction, will, if placed on a different face, and measuring 2, 3, 1 inches respectively in the three directions; and supported to stand on a corner the corresponding measures can be varied within wide limits. The length, the breadth and the thickness of the brick have no separate existence; and what exists is a totality of the three. How the totality is broken up by the senses depends upon the circumstances of the individual. Going a step higher, space and time also have no separate existence; what exists is a totality which the mind is not able to grasp as a whole, and consequently the sense breaks it up into four convenient parts, different for different observers. This is not a metaphysical speculation; this is a theory forced upon us by unexpected phenomena displayed by rapidly moving bodies, the details of which are too technical to be described here.

Two objections at once strike us: How can time and space be similar if we do not perceive them in the same way? And how is it that which in space we can move up and down and sideways, in time we can only move forward, and we do so move whether we want to or not? To see that the objections are not valid, we revert to our initial sense-impressions and note that the three dimensions of space are *not* perceived by the same senses. Two of the dimensions become known to us through the eye, when our sight is obstructed; the third dimensions—thickness or distance,—has to be discovered with the help of sight, touch, and muscular

effort and binocular vision. The fact is that the three space dimensions are *not* interchangeable for *all* purposes, and a mile up-hill is *not* the exact equivalent of a mile on the level. The difference is due to the structure of the human body, and it is only when by a mental effort we eliminate part of the personality of the observer that we can say that space is homogeneous. The perceptions of time and of space come to us through different sets of sense-impressions; but the discussion above shows that *that* does not necessarily indicate an essential difference, As regards the monotony and the apparent inevitability of movement in time, the explanation may be the same—our physical and mental circumstances.

We are now in a position to give some account of the picture of the universe that can be built out of Einstein's equations; this picture is not Einstein's, who steadily refuses to go beyond his equations and symbols; but the human mind is not satisfied with merely x and y and insists on knowing whether the symbols have an interpretation in the material universe. The physicist gives the interpretation, but he knows that it is only plausible being based on necessarily incomplete data, that the picture of the universe that emerges is only an approximation to reality,—if there be a reality,—and that the picture may have to be discarded when further data accumulate. Nevertheless, he provides the picture as a help to (his own and other people's) clear thinking; and the picture he gets out of Einstein is something like this. "There is a world-stuff spread in four dimensions, with a three dimensional continuum (our universe) as its section or boundary, much as the earth is stuff spread in three dimensions with a two dimensional surface as its boundary. The continuum, like the surface of the earth, is uneven, the unevenness of the former being of course in four dimensions and not in three like those of the earth. This unevenness we call matter. Our momentary view of the universe changes as if each point of it moved in the straightest line, and we call the change the passage of time". Why the universe seems to change and why the movement in the straightest line, we do, now know.

The theory leads to the result that space, time and matter are essentially one, but it leaves electricity out as a separate entity. Weyl has framed a new theory, based on a geometry more general than that of Riemann, and has included electricity among the property of space, or rather of the world-stuff. But we have not reached the goal in our search for a scientific monism. The entities we call life and mind still stand out of the mechanical scheme. The experimental method is, however, being applied rapidly to biology and psychology, and the next two hundred years might see the end of the isolation.

Prof. Devi Dyal's concluding Speech.

On behalf of the Punjab University Mathematical Association, it is my pleasant duty to offer you our heartiest thanks for coming to attend the Conference. I know that some of you have taken the trouble to travel over 2000 miles to contribute to the success of the gathering. It has indeed been a great pleasure and privilege to meet so many distinguished delegates from all parts of my country. It was only towards the end of February that the Indian Mathematical Society accepted our invitation and decided on holding its third Conference at Lahore. As soon as we learnt of it, we approached the Vice-Chancellor, the Chief Justice and both the Ministers; their sympathy, co-operation and financial help were generously given. In fact, our popular Vice-Chancellor, without whose active support, I fear, it would have been almost impossible to hold the Conference, moved the Syndicate to sanction a handsome donation of Rs. 250 to meet a part of the expenses. You have to-day befittingly recorded a resolution expressing your grateful thanks to him and to the Syndicate of the University.

We are equally grateful to H.E. the Governor who was pleased to accept our invitation to open the Conference. It is needless to say that the Conference felt itself highly honoured by His Excellency's presence at the opening session and flattered by his kindly welcome. Our best thanks are also due to the Hon'ble Khan Bahadur Mian Fazal Hussein, who kindly agreed to be the Chairman of the Reception Committee and to the Hon'ble Sardar Bahadur Sardar Sunder Singh Majithia who, in the unavoidable absence of the Educational Minister, performed his duties with cheerfulness, and invited the members to a splendid tea-party at his bungalow.

Gentlemen, it is thus evident that the success of the Conference has been due to the willing and cheerful co-operation of all. If we have failed to make satisfactory arrangements for your comforts, I request you to forgive us; for, this could not be due to lack of thought. I once more thank you for kindly visiting our province and infusing in us a spirit of mathematical study and research. You have shown us how wide is the field for work. Let me assure you that your visit will long live in our memory and our Association will never forget the honour done to it by your presence.

An Extension of Feuerbach's Theorem

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Introductory.

IN the post-script to the paper on "An Extension of Feuerbach's Theorem" read by me at the last Conference of the Indian Mathematical Society and published in the Journal of the above Society for December 1919, I stated as below :—

The following property suggested by Feuerbach's Theorem may be found interesting. My proof is not complete :

"Of the system of pedal-contact circles, each circle touches four circles which latter is also touched by three more circles of the system."

The proof of the above property is presented in this paper. It is based on the following theorems :—

(i) Of the system of pedal circles of pairs of isogonal conjugates of lines passing through any fixed point, the envelope of the radical axes of any circle and the remaining circles of the system is a conic.

(ii) Isogonal conjugate points on any four lines through the fixed point referred to in (i), which taken two by two belongs to three definite systems of involution, form a pair of Steinerian quadruples.*

The paper is divided into three sections. The first gives the properties of pedal-contact circles which are required for the purpose of this paper. The second section deals with the conic referred to in (i) above, and contains the proof of the main theorem. In the last section, a method of associating each tetrad of circles mentioned in the main theorem with a definite point is given and the relation between the two tetrads of tangent circles is established.

SECTION I.

The Pedal-Contact Circle.

§ 1. The pedal-contact circle is defined to be the pedal circle of a point P whose pedal triangle is in perspective with the triangle of reference ABC.[1] The join of P and its isogonal conjugate P′ passes through

* This result is stated on page 225, Vol, XI, Journal of the Indian Mathematical Society.

[1] Cf. Author's paper, Journal of the Indian Mathematical Society *loc. cit.*

the orthocentre of the antimedial triangle of ABC.[2] The locus of P is the self-isogonal cubic which passes through A, B, C, the circum-centre, the orthocentre, the in-and the ex-centres of ABC and cuts the sides of the triangle ABC again in the meets of AP, BP, CP, with BC, CA, AB, and the circum-circle of ABC in the points diametrically opposite to A, B, C. The circum-centre of ABC is also the centre of the cubic and its asymptotes are the perpendicular bisectors of the sides of ABC.[3] Its trilinear equation is

$$(\cos A - \cos B \cos C)\, \alpha(\beta^2 - \gamma^2) + (\cos B - \cos C \cos A)\, \beta(\gamma^2 - \alpha^2) + (\cos C - \cos A \cos B)\, \gamma\, (\alpha^2 - \beta^2) = 0.$$

§ 2. Let DEF be the pedal triangle of P and D′E′F′, that of P′, the isogonal conjugate of P. Let EF, and E′F′ intersect in X and EF′ and E′F in X′. If Y, Y′ and Z, Z′ be similar points obtained from the remaining sides of DEF and D′E′F′, the following points are collinear :—

(1) X′, Y′, Z′. (2) X′, Y, Z. (3) Y′, Z, X. (4) Z′, X, Y.

The lines of collinearity are evidently the pascal lines of the hexagons DE′FD′EF′, etc. This holds good for any conic cutting the sides of A, B, C in D, D′; E, E′; F, F′. The lines AX, BY, CZ are concurrent at a point π which is the pole of X′Y′Z′ with respect to the conic. Further if AD, BE, CF meet in a point Q, AD′ BE′ CF′ will meet in another point Q′; the pascal line X′Y′Z′ passes through Q and Q′ and the pascal lines X′YZ, XY′Z and XYZ′ respectively pass through A, B, C. The triangle XYZ will now be self-conjugate with respect to the conic.

The above results for conics are well-known. In applying them to the case of the pedal circle, we note that for any pedal circle the pascal line X′Y′Z′ passes through the isogonal conjugate points P and P′ and hence is a diameter of the pedal circle. The lines AX, BY, CZ are parallel and perpendicular to the line PP′X′Y′Z′,[4] the point π being now the point at infinity in direction at right angles to this line.

§ 3. It is due to the concurrency of AD, BE, CF, that the triangle XYZ becomes self-conjugate with respect to the circle. Hence we have :

(1) In the case of any circle DD′EE′FF′, the triangle XYZ is in perspective with ABC.

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[2] Cf. Author's paper on Double Lines, page 125, Vol. XII, Journal of the Indian Mathematical Society.

[3] The geometrical treatment of this cubic forms the subject of a paper under preparation.

[4] For an elementary proof, cf. author's note on 'The Pedal Circle,' page 145 Vol. IV, Journal of the Indian Mathematical Society.

(2) If the circle is a pedal circle, the centre of perspective is at infinity.

(3) If the circle is a contact circle, the triangle is self-conjugate with respect to the circle.

The last of the results stated above is important. From this it follows that

(4) DX, EY, FZ, concur at a point W on the circle, and D'X, E'Y, F'Z concur at a point W' also on the circle.

(5) The line WW' is the triangular polar of π with respect to ABC.

The triangle XYZ is the common diagonal triangle of the quadrilaterals DEFW and D'E'F'W'. If DW and DW' cut YZ in R and R', the ranges (DX, RW) and (D'X, R'W') are harmonic. Therefore DD', *i.e.*, BC, RR', *i.e.*, YZ and WW' are concurrent. Similarly WW' passes through the meets of CA, AB, with ZX, XY respectively, etc.

(6) XYZ is the diagonal triangle of the quadrilateral formed by the sides of ABC and the line WW'.

§ 4. The following properties are special to the pedal-contact circle :—

(7) The points X, Y, Z, lie on the sides of the medial triangle of ABC.[5]

(8) The conic circumscribing the triangle ABC, and passing through Q and its isotomic conjugate is a rectangular hyperbola.[6]

(9) The points W, W' referred to in (4) are the centres of the rectangular hyperbolas ABCQ and ABCQ' and therefore lie on the nine-points circle of ABC.[7]

The main result of this section which is required for the proof of the theorem of this paper may be stated as below :—

The triangle XYZ is self-conjugate with respect to the pedal circle of P and also self-conjugate with respect to the in-conics of the quadrilateral formed by the sides of ABC and WW' (which is the radical axis of the nine-point circle and the pedal circle).

SECTION II.

The Proof of the Theorem.

§ 5. Let us consider the pedal circles of points on the self-isogonal cubic,

$$p\alpha(\beta^2-\gamma^2)+q\beta(\gamma^2-\alpha^2)+r\gamma(\alpha^2-\beta^2)=0. \quad \ldots \quad (1)$$

[5], [6], [7].—These results are taken from the paper referred to in foot-note ([3]).

The radical axis of any fixed circles of the system and any of the remaining circles touches a conic. In order to establish this, it is sufficient to prove that through any arbitrary point, two and not more than two of these radical axes pass. Given any point T, the power of T with respect to S is known, and the locus of a point whose pedal circle has the same power with respect to T is another self-isogonal cubic of the form

$$l\alpha\,(\beta^2 + \gamma^2) + m\beta(\gamma^2 + \alpha^2) + n\gamma(\alpha^2 + \beta^2) + 2s\,\alpha\beta\gamma = 0 \;\ldots \quad (2)$$

The cubics (1) and (2) intersect in 9 points of which three are the vertices of the triangle of reference ABC. The remaining six arrange themselves into three pairs of isogonal conjugate points of which one pair, namely, the points whose pedal circle is the fixed circle S, is known. Hence there are two pairs of isogonal conjugate points on (1) whose pedal circles with respect to T has the given power; that is to say, only two radical axes pass through T. The cubic (1) is the locus of a point which lies in a line with its isogonal conjugate and a given point (p, q, r). Hence we have the following results :—

(10) Of the system of pedal circles of pairs of isogonal conjugate points whose join passes through a fixed point, the envelope of the radical axes of any given circle and the remaining circles of the system is a conic [8].

This conic may be briefly denoted as the envelope-conic associated with the given pedal circle.

(11) A given pedal circle of the system touches *four* pedal circles of the same system.

This is evident from the above result. The tangents at the points of contact are the common tangents of the given pedal circle and the envelope-conic referred to in (10).

§ 6. Let J be the fixed point in (1) and through it draw a fixed line L and a variable line M. Since the radical axis of the pedal circles, say (L) and (M) of isogonal coujugate points on L and M touches a conic, there is a one-to-one correspondence between the line M and the corresponding radical axis. Hence we see that

(12) Lines through J correspond anharmonically with the radical axis of (L) and (M) and with their points of contact on the envelope-conic.

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[8] The envelope of the radical axis of the pedal circle of the fixed point J and the remaining circles of the system is the orthopolar ellipse of J. I have got a more direct proof of the result (10) by the anharmonic properties of conics. This incidentally determines the position of the centre of the envelope-conic,

In the general case when J is an arbitrary point, a pedal circle (1) is touched by four pedal circles (M). Each of these latter is touched by 4 pedal circles of which the circle (L) is one. This result is not of much interest. Suppose that when J is a particular point we are able to prove that the isogonal conjugate points on the four lines M—say M, M_1, M_2, M_3 whose pedal circles touch any circle (L) of the system, form a pair of Steinerian quadruples. In this case, the lines L, L_1, L_2, L_3 corresponding to the circles (L), $(L)_1$, $(L)_2$, $(L)_3$, which touch one of the pedal circles (M) are such that the isogonal conjugate points on them form also a pair of Steinerian quadruples. When one point of a Steinerian quardruple is given, the remaining three are uniquely determined, the circles (L), $(L)_1$, $(L)_2$, $(L)_3$, are repeated for each of the 4 circles (M), $(M)_1$, $(M)_2$, $(M)_3$. Hence each of the four circles (L) will touch each of the four circles (M).

In order to establish the theorem, it is sufficient to prove that :—

> When J is the orthocentre of the anti-medial triangle of ABC, the lines M, M_1, M_2, M_3, corresponding to the line L are such that the isogonal conjugate points on them form a pair of Steinerian quadruples.

§ 7. The condition that the isogonal conjugate points on four lines M, M_1,M_2,M_3, passing through J form a pair of Steinerian quadruples is satisfied when the rectangular hyperbola through J intersects on these lines in points forming a qudrangle having ABC as the diagonal triangle, *i.e.*, when the following systems are in involution :—

$(MM_1,\ M_2M_3,\ JB,\ JC)$ ... ... (A)

$(MM_2,\ M_3M_1,\ JC,\ JA)$ ... ... (B)

$(MM_3,\ M_1M_2,\ JA,\ JB)$ ... ... (C)

Lines drawn from J and through the points on the cubic forming a Steinerian quadruple taken two by two, belong to the above three systems, which may be denoted as the (A), (B), (C) involutions.

The (A) system is determined as follows :—

> Take any point U on the cubic. Let AU produced meet the cubic again in U_1. Then JU and JU_1 are conjugate rays of the (A) system.

Similarly for the (B) and (C) systems.

§ 8. When J is the orthocentre of the anti-medial triangle of ABC, the cubic passes through H, the orthocentre of ABC, and AH, BH, CH are parallel to the asymptotes of the cubic. If J_a J_b,J_c, denote the perpendiculars on the sides of ABC and J_n denote the Euler line JONH (O circumcentre, N-nine-point centre of ABC), the rays (J_nJ_a), (J_nJ_b), (J_nJ_c), are conjugate rays of the (A), (B), (C) systems respectively, as also the rays (J_bJ_c), (J_cJ_a), (J_aJ_b). Hence the conditions under consideration may be written as :—

$$(MM_1,\ M_2M_3,\ J_nJ_a,\ J_bJ_c) \quad \ldots \quad \ldots \quad (A)$$
$$(MM_2,\ M_3M_1,\ J_nJ_b,\ J_cJ_a) \quad \ldots \quad \ldots \quad (B)$$
$$(MM_3,\ M_1M_2,\ J_nJ_c,\ J_aJ_b) \quad \ldots \quad \ldots \quad (C)$$

The whole question has been reduced to one of showing that the M-lines taken two and two are conjugate rays of the pencils in involution determined by the four lines, viz. the perpendiculars from J on the sides of ABC and the Euler line OH of ABC, taken two and two. This is the form best suited for our purpose.

§ 9. The envelope-conic associated with the line L touches the line circles of the pedal-contact system, *viz.*, the side of ABC, and the radical axis WW′ of the pedal circle of isogonal conjugate points on L, which we have denoted by (L), and the nine-point circle. Let α, α_1, α_2, α_3 be the points of contact of the envolope-conic with WW′ and the sides of ABC. By the result stated at the end of the first section, we see that the triangle XYZ is self conjugate with respect to the envolope-conic. Therefore XYZ is the diagonal triangle of the quadrangle $\alpha\alpha_1\alpha_2\alpha_3$. Let the points of contact of the (L) circle with the M circles be β, β_1, β_2, β_3. The common tangents at these points touch the envelope conic. If γ, γ_1, γ_2, γ_3 are the points of contact on the envelope-conic, we see by a known theorem in conic sections, that XYZ is the diagonal triangle of each of the quadrangles $\beta\beta_1\beta_2\beta_3$ and $\gamma\gamma_1\gamma_2\gamma_3$, since it is self-conjugate with respect to the circle (L) and the envolope-conic. Thus we see that the five quadrangles DEFW, D′E′F′W′, $\alpha\alpha_1\alpha_2\alpha_3$, $\beta\beta_1\beta_2\beta_3$, $\gamma\gamma_1\gamma_2\gamma_3$, have a common diagonal triangle XYZ.

Consider the α- and γ-points on the envelope-conic. It is seen that the lines

$$\alpha\alpha_1,\ \alpha_2\alpha_3,\ \gamma\gamma_1,\ \gamma_2\gamma_3, \text{ intersect in X,} \quad \ldots \quad (1)$$
$$\alpha\alpha_2,\ \alpha_3\alpha_1,\ \gamma\gamma_2,\ \gamma_3\gamma_1, \text{ intersect in Y,} \quad \ldots \quad (2)$$
$$\alpha\alpha_3,\ \alpha_1\alpha_2,\ \gamma\gamma_3,\ \gamma_1\gamma_2, \text{ intersect in Z.} \quad \ldots \quad (3)$$

Therefore the points in (1), (2), (3) form three involutions. According to the correspondence stated in § 6 of this section, the α-points correspond anharmonically with WW′ and the sides of ABC and these latter with the lines J_n, J_a, J_b, J_c. The γ-points correspond with the M-lines. It is evident that the pencils

$$(MM_1,\ M_2M_3,\ J_nJ_a,\ J_bJ_c)$$
$$(MM_2,\ M_3M_1,\ J_nJ_b,\ J_cJ_a)$$
$$(MM_3,\ M_1M_2,\ J_nJ_c,\ J_aJ_b)$$

form three involutions. Hence the condition obtained in § 8 of this Section is satisfied and consequently the theorem is established.

SECTION III.

§ 10. In the preceding section, we have shown that a given pedal-contact circle (L) is touched by four similar circles (M), which latter are

touched by three more circles associated with the given circle. We shall now examine the relation between the (L) and (M) circles more closely.

The pedal-contact circles may be arranged into two sets, each set containing groups of four pedal circles of points forming a Steinerian quadruple, a group of the first set corresponding with that group of the second whose members touch the circles of the former group. Any circle of a group of either set determines not only the other members belonging to the group, but also the circles of the corresponding group of the other set. Hence corresponding groups may be taken to belong to either set, that is to say, the correspondence is mutual. If we can establish a (1, 1) correspondence between each group and points on a line, the corresponding points will form an involution. This may be done as follows :—

Let J be the point (l, m, n), where $l = \cos A - \cos B \cos C$, $m = \cos B - \cos C \cos A$, $n = \cos C - \cos A \cos B$. The locus of isogonal conjugate points on lines through J lie on the cubic,

$$l\alpha(\beta^2 - \gamma^2) + m\beta(\gamma^2 - \alpha^2) + n\gamma(\alpha^2 - \beta^2) = 0.$$

If P be any point on this cubic, its associated points of the Steinerian quadruple are given by the following construction :—

Let JP meet in R, the first polar of J with respect to the cubic, *viz.*, the rectangular hyperbola.

$$l^2(\beta^2 - \gamma^2) + m^2(\gamma^2 - \alpha^2) + n^2(\alpha^2 - \beta^2) = 0 \qquad \ldots \quad (1)$$

and AQ, BQ, CQ intersect the rectangular hyperbola in Q_1, Q_2, Q_3. If P' is the isogonal conjugate of P, then AP', BP', CP' intersect respectively JQ_1, JQ_2, JQ_3, in the required points, say P_1, P_2, P_3, lying on the cubic. It will be seen therefore that the pedal circles of P, P_1, P_2, P_3 are associated respectively with the lines JQ, JQ_1, JQ_2, JQ_3. The rectangular hyperbola (1) is self-conjugate with respect to the triangle ABC. The co-ordinates of the Q-points may be taken to be $(x, \pm y, \pm z)$, satisfying the equation $\Sigma l^2(y^2 - z^2) = 0$.

The four Q points may evidently be associated with a single point R (px^2, qy^2, rz^2), where p, q, r are arbitrary constants.

The locus of R is the line

$$l^2\left(\frac{\beta}{q} - \frac{\gamma}{r}\right) + m^2\left(\frac{\gamma}{r} - \frac{\alpha}{p}\right) + n^2\left(\frac{\alpha}{p} - \frac{\beta}{q}\right) = 0;$$

i.e.
$$\frac{\alpha}{p}(m^2 - n^2) + \frac{\beta}{q}(n^2 - l^2) + \frac{\gamma}{r}(l^2 - m^2) = 0. \qquad \ldots \quad (2)$$

The nine-point-circle and the sides of the triangle form a group for which the Q-points are $-\left(\frac{1}{a}, \pm\frac{1}{b}, \pm\frac{1}{c}\right)$, and R is $\left(\frac{p}{a^2}, \frac{q}{b^2}, \frac{r}{c^2}\right)$.

The in- and the ex-circles form the corresponding group for which the

Q-points are $(1, \pm 1, \pm 1)$ and the R′ is (p, q, r). R′ obviously lies on (2) and R also lies on (2) by virtue of the relation

$$\frac{l}{\cos A - \cos B \cos C} = \frac{m}{\cos B - \cos C \cos A} = \frac{n}{\cos C - \cos A \cos B}.$$

To determine the involution to which R, R′ belong, it is necessary to find two more such points. For the group to which the pedal circle of J belongs R is (pl^2, qm^2, rn^2), which evidently lies on (2). To find the corresponding group, bisect internally and externally the angles BJC, CJA, AJB. If the bisectors meet the sides BC, CA, AB in X, X′; Y, Y′; Z, Z′ the pedal circles of isogonal conjugate points on the perpendiculars from J to X′Y′Z′, X′YZ, Y′ZX, Z′XY, from the corresponding group [9]

The co-ordinates of R′ corresponding to this group are found to be

$$\frac{p(2 \cos^2 B + 2 \cos^2 C - \sin^2 A)}{\sin^2 A}, \text{ \&c., \&c.}$$

The involution is completely determined by the conjugate points

$$(p, q, r); \left(\frac{p}{\sin^2 A}, \frac{q}{\sin^2 B}, \frac{r}{\sin^2 C}\right);$$

and $$(pl^2, qm^2, rn^2); \left\{\frac{p(2 \cos^2 B + 2 \cos^2 C - \sin^2 A)}{\sin^2 A}, \ldots\ldots \right\}.$$

Since these points are collinear, we have

$$\frac{p + \dfrac{kp}{\sin^2 A}}{pl^2} = \frac{q + \dfrac{kq}{\sin^2 B}}{qm^2} = \frac{r + \dfrac{kr}{\sin^2 C}}{rn^2}, \qquad \ldots \quad (1)$$

and $$\left(p + \frac{k'p}{\sin^2 A}\right) = \frac{p}{\sin^2 A}(2 \cos^2 B + 2 \cos^2 C - \sin^2 A)$$

$$= \left(q + \frac{k'q}{\sin^2 B}\right) = \frac{q}{\sin^2 B}(2 \cos^2 C + 2 \cos^2 A - \sin^2 B)$$

$$= \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2), we find that

$$k = -\frac{\sin^2 A \sin^2 B \sin^2 C}{2 \cos A \cos B \cos C},$$

and $$k' = -4 \cos A \cos B \cos C;$$

ann hence the constant of involution is given by

$$kk' = 2 \sin^2 A \sin^2 B \sin^2 C.$$

The foci of the involution are real. Corresponding to each focus, there are therefore four lines through J such that the four pedal circles of isogonal conjugate points on the lines touch one another.

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[9] *Vide*: Author's note on page 105, Vol, IV, Journal of the Ind. Math. Soc,

SOLUTIONS.

Question 927.

(M. K. KEWALRAMANI)—A B C D E is a polygon whose sides taken in order are a, b, c, d, e...... If the angles which the sides b, c, d, e...make with the positive direction of the side a, be respectively called ab, ac, ad, ae......prove that

$$a^n = \sum \frac{n!}{r!\,s!\,t!\,\ldots} b^r c^s d^t \left(\cos \left\{ r.ab + s..ac + t.ad + \ldots \right\} \right)$$

where n is a positive integer and r, s, t are also positive integers including zero but such that $r + s + t + \ldots = n$ and Σ denoting summation of like expressions when r, s, t ... are taken in their entirety subject to the above restriction; of course the right-hand side a not occurring.

Solution by N. Sankara Aiyar.

It can easily be seen that

a = real part of $\Sigma\, b\,(\cos ab + i \sin ab)$ the summation extending to the $(n-1)$ sides but a. Hence a^n = real part of $\{\Sigma b\,(\cos ab + i \sin ab)\}^n$. The typical term in the expansion of this expression is given by the multinomial theorem in the form

$$\frac{n!}{r!\,s!\,t!\,\ldots} b^r c^s d^t \ldots (\cos ab + i \sin ab)^r (\cos ac + i \sin ac)^s (\cos ad + i \sin ad)^t \ldots \text{ where } r + s + t + \ldots = n.$$

i.e., $$\frac{n!}{r!\,s!\,t!\,\ldots} b^r c^s d^t \ldots (\cos r.ab + i \sin r.ab) \times (\cos s\,ac + i \sin s.ac)(\cos t.ad + i \sin t.ad) \times \ldots$$

i.e., $$\frac{n!}{r!\,s!\,t!\,\ldots} b^r c^s d^t \ldots [\cos\{r.ab + s.ac + t.ad + \ldots\} + i \sin\{r.ab + s.ac + t.ad + \ldots\}].$$

By summing up the real parts of all the terms we get the required result.

Question 929.

(S. MALHARI RAO):—Express 2595600 as the product of two factors such that the sum of the factors of either factor may be equal to the other factor.

Solution by N. Sankara Aiyar.

The sum of the factors of any number of the form $\alpha^p \beta^q \gamma^r \ldots =$

$$\frac{(\alpha^{p+1} - 1)(\beta^{q+1} - 1)\ldots}{(\alpha - 1)(\beta - 1)\ldots}$$

If AB = 2595600 = $2^4 \times 5^2 \times 3^2 \times 7 \times 103$ and 103 is a factor of A, 104 should be a factor of B. This is clearly impossible. If however the question is :—'the sum of the factors of either factor may be the same,' we get a solution. 3^2 should be a factor of B. So also should 7 be. Similarly 2^4 may be a factor of A or B and then 5^2 will be a factor of B or A. The two solutions are thus $5^2 \times 103$, $2^4 \times 7 \times 3^2$ and $2^4 \times 103$, $5^2 \times 7 \times 3^2$.

Question 1121.

(N. Durai Rajan):—Constant lengths a are measured along the tangents to a curve. If the locus of the extremities be a circle of radius a, show that the curve is the inverse of an involute of a circle of radius $2a$, the radius of inversion being equal to $2a$.

Solution by Martyn M. Thomas and others.

Let a length PQ ($= a$) be measured along the tangent at P to the curve required.

Then, if P be (x, y), Q will be $(x+a\cos\psi, y+a\sin\psi)$, where ψ is the angle made by the tangent with the x-axis.

Since Q lies on a circle of radius a,

$$(x+a\cos\psi)^2 + (y=a\sin\psi)^2 = a^2.$$

$$\therefore \quad x^2+y^2+2a\,(x\cos\psi + y\sin\psi) = 0$$

$$\therefore \quad x^2 + y^2 + 2a\left(x\frac{dx}{ds} + y\frac{dy}{ds}\right) = 0.$$

$$\therefore \quad -\frac{ds}{a} = \frac{2(xdx + ydy)}{x^2 + y^2} = d\,[\log(x^2+y^2)] = d\,[2\log r].$$

$$\therefore \quad -\frac{s}{a} = 2\log r + \text{const.}$$

Differentiating with respect to θ,

$$-\frac{ds}{d\theta} = 2a\,.\,\frac{1}{r}\frac{dr}{d\theta}.$$

Squaring $r^2 + \left(\frac{dr}{d\theta}\right)^2 = \frac{4a^2}{r^2}\left(\frac{dr}{d\theta}\right)^2.$

$$\therefore \quad \frac{dr}{d\theta} = \frac{r^2}{\sqrt{4a^2 - r^2}},$$

which is the polar differential eqn. of the reqd. curve.

Now, the (p, r) equation of an involute of a circle of radius $2a$ is

$$r^2 = p^2 + 4a^2. \quad \dots \quad \dots \quad \dots \quad (1)$$

Since for inverse curves, $rr' = k^2$ and $\frac{p}{r} = \frac{p'}{r'}$, change

$$r \text{ to } \frac{k^2}{r} \text{ and } p \text{ to } \frac{k^2 p}{r^2}$$

in eqn. (1).

Thus the inverse of the involute of the circle of radius $2a$ is

$$\frac{k^4}{r^2} = \frac{k^4 p^2}{r^4} + 4a^2.$$

Putting $k = 2a$, $r^2 = p^2 + \frac{r^4}{4a^2}$.

$$\therefore \quad \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{p^2} = \frac{4a^2}{r^2(4a^2 - r^2)}.$$

$$\therefore \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{4a^2 - r^2}.$$

Hence $$\frac{dr}{d\theta} = \frac{r^2}{\sqrt{4a^2 - r^2}},$$

which is the same as the polar differential equation found above.

Question 1124.

(G. S. M.) :—Prove that

$$1 + (r, 1)\frac{n}{m} + (r, 2)\frac{n(n-1)}{m(m-1)} + \dots \text{ to } (n+1) \text{ terms}$$
$$= \frac{(m+1)(m+2)\dots(m+r)}{(m-n+1)(m-n+2)\dots(m-n+r)}$$

where $(r, 1)$, $(r, 2)$ ... are the figurate numbers of the rth order.

Solution by K. J. Sanjana, V. M. Gaitonde and others.

The given left side may be written more generally thus :—

$$1 + {}^nC_1\frac{r}{m} + {}^nC_2\frac{r(r+1)}{m(m-1)} + \dots,$$

where n is a positive integer, but r, m are any numbers.

We have by Van der Monde's theorem,

$$(x+y)_n = x_n + {}^nC_1 x_{n-1} y_1 + {}^nC_2 x_{n-2} y_2 + \dots \text{ to } (n+1) \text{ terms}.$$

Hence $$\frac{(x+y)_n}{x_n} = 1 + {}^nC_1\frac{y}{x-n+1} + {}^nC_2\frac{y(y-1)}{(x-n+1)(x-n+2)} + \dots,$$

where x and y are any numbers. Put $x = -m + n - 1$, $y = -r$, and we get

$$\frac{(-m+n-1-r)_n}{(-m+n-1)_n} = 1 + {}^nC_1\frac{-r}{-m} + {}^nC_2\frac{-r(-r-1)}{-m(-m+1)} + \dots$$
$$= 1 + r \cdot \frac{n}{m} + \frac{r(r+1)}{1 \cdot 2}\frac{n(n-1)}{m(m-1)} + \dots \text{ to } (n+1) \text{ terms}.$$

Also $$\frac{(-m+n-1-r)_n}{(-m+n-1)_n} = \frac{(m+r)(m+r-1)\dots(m+r-n+1)}{m(m-1)\dots(m-n+1)}$$

where there are n factors both in the numerator and the denominator.

When r is a positive integer, $\frac{r}{1}$, $\frac{r(r+1)}{1 \cdot 2}$, ... are the second, third, ... figurate numbers of the rth order; and we can cancel factors above and below from $m \equiv (m + r - r)$ to $(m - n + r + 1)$, and thus obtain the given result.

QUESTIONS FOR SOLUTION.

1173. (S. KRISHNASWAMI AIYANGAR):—If $w_1 + w_2 + w_3 = 0$, show with the usual notation in elliptic functions $s_1c_1d_1 + s_2c_2d_2 + s_3c_3d_3$

$$= -(s_1s_2s_3)^{-3}[k^4 - (s_1s_2s_3)^2(l - c_1c_2c_3)k^2 - 2(s_1s_2s_3)]^2.$$

1174. (S. KRISHNASWAMI AIYANGAR):—If $w_1 + w_2 + w_3 = K$, show that

$$s_1c_1d_1 + s_2c_2d_2 + s_3c_3d_3$$

$$= \left(\frac{2-k^2}{k'}\right) c_1c_2c_3 + 3k^2 s_1s_2s_3\, c_1c_2c_3 - k^4.$$

1175. (M. K. KEWALRAMANI):—Shew that the area of any part of the curve $y(x + m)(x - m) = n$, depends on the rectification of a certain parabola, and find out the general form of curves whose areas will be constant multiples of the corresponding parabolic length, the same parabola being employed in each case. Construct the above curve geometrically by means of the parabola.

1176. (A. NARASINGA RAO):—In Basset's "*Cubic and Quartic Curves*"* it is stated that the orthoptic locus of an Epi-cycloid is an Epi-trochoid. Show that Basset's Epi-trochoid is but part of the complete locus, which consists of $\frac{m+2}{2}$ such epi-trochoids, or $\frac{m+1}{2}$ trochoids and a circle repeated $\frac{m+1}{2}$ times, according as m (the ratio of the fixed to the rolling circle) is even or odd.

Prove also that the orthoptic locus splits up into algebraically distinct curves, whenever a rational curve of class λ has $\tan\frac{\psi}{\lambda}$ as a possible parameter, and that this happens only when all the tgts. from the circular points to the curve, coalesce.

1177. (A. NARASINGA RAO):—If the tangents at P and P^1 of a plane curve meet at right angles at O, show that the curvature of the orthoptic locus at O is $\frac{2c^2 - Pr^1 - P^1r}{c^3}$ where P and P^1 are the radii of curvature at P and P^1, $PO = r$, $P^1O = r^1$ and $PP^1 = c$.

1178. (A. NARASINGA RAO):—Trace the curve whose equation in intrinsic co-ordinates is

(i) $S = e^{a\psi} \sin b\psi$.

(ii) $S = e^{a_1\psi} \sin b_1\psi + e^{a_2\psi} \sin b_2\psi$.

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* Pages 213 and 214.